

# ULTRAPRODUCTS AND METASTABILITY

JEREMY AVIGAD AND JOSÉ IOVINO

**ABSTRACT.** Given a convergence theorem in analysis, a model-theoretic compactness argument can often be used to show that there is a uniform bound on the rate of metastability. We illustrate this with three examples from ergodic theory.

## 1. INTRODUCTION

Convergence theorems in analysis are often disappointingly nonuniform. For example, Krengel [27] has shown, roughly speaking, that even if one fixes an ergodic measure preserving system, the convergence of averages guaranteed by the mean ergodic theorem can be arbitrarily slow. The goal of this note is to show that even in such cases, a straightforward compactness argument can often be used to establish a weaker uniformity, namely, the existence of uniform bounds on the rate of metastable convergence.

If  $(a_n)_{n \in \mathbb{N}}$  is a sequence of elements in a metric space  $(X, d)$ , saying that  $(a_n)$  is Cauchy is equivalent to saying that for every  $\varepsilon > 0$  and function  $F : \mathbb{N} \rightarrow \mathbb{N}$ , there is an  $n$  such that  $d(a_i, a_j) < \varepsilon$  for every  $i, j \in [n, F(n)]$ . Think of  $F$  as trying to disprove the convergence of  $(a_n)$  by finding intervals where the sequence fluctuates by more than  $\varepsilon$ ; the  $n$  asserted to exist foils  $F$  in the sense that the sequence is remains  $\varepsilon$ -stable on  $[n, F(n)]$ . We will call a bound on such an  $n$ , depending on  $F$  and  $\varepsilon$ , a *bound on the rate of metastability*.

The arguments below show that, in many convergence theorems, there is a bound on the rate of metastability that depends on only a few of the relevant parameters. All that is required is that the class of structures in question, and the hypotheses of the theorem, are preserved under a certain model-theoretic ultraproduct construction in which these parameters remain fixed. This requirement can be formulated in syntactic terms, by asserting that the relevant hypotheses and axioms can be put in a certain logical form. We spell this out in Section 2, which summarizes the necessary background on ultraproducts in analysis. Section 3 illustrates the method with three examples from ergodic theory.

Metastability has proved useful in ergodic theory and ergodic Ramsey theory [13, 41, 42]; see also [43, Sections 1.3–1.4], and [4, 3, 24, 22, 23, 25, 26] for various instances of metastability in analysis. Tao [40] relates the existence of uniform bounds on the rate of metastable convergence of a collection of sequences to non-standard convergence statements in a corresponding ultraproduct.

Sometimes stronger uniformities are available than the ones we consider here, in the form of square-function inequalities (for example, as in Jones, Ostrovskii, and

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Rosenblatt [19]), or bounds on the number of fluctuations; see the discussion in [5]. Bergelson et al. [8] explore aspects of uniformity in ergodic theory and ergodic Ramsey theory; most of the methods there rely on specific combinatorial features of the phenomena at hand.

The methods developed here complement proof-theoretic methods developed by Kohlenbach and collaborators, e.g. in [21, 12]. Roughly, those methods provide “metatheorems” which show that when a statement with a certain logical form is derivable in a certain (fairly expressive) axiomatic theory, certain uniformities always obtain. The arguments we present here replace derivability in an axiomatic system with closure under the formation of ultraproducts. Indeed, it seems likely that such arguments can be used to establish general metatheorems like the ones in [21, 12], by considering ultraproducts of models of the axiomatic theories in question.

It is worth noting that although the methods we describe here can be used to establish the existence of a very uniform bound, they give no explicit quantitative information at all, nor even show that it is possible to *compute* such a bound as a function of  $F$  and  $\varepsilon$ . In contrast, the proof-theoretic techniques provide ways that such information can be “mined” from a specific proof. If one is primarily interested in uniformity, however, the methods here have the virtue of being easy to understand and apply.

## 2. ULTRAPRODUCTS OF BANACH SPACES

In this section we review standard ultraproduct constructions in analysis; see [7, 15, 16, 48] for more details.

Let  $I$  be any infinite set, and let  $D$  be a nonprincipal ultrafilter on  $I$ . (Below, we will always take  $I$  to be  $\mathbb{N}$ .) Any bounded sequence  $(r_i)_{i \in I}$  of real numbers has a unique limit with respect to  $D$ ,  $r = \lim_{i, D} r_i$ ; this means that for every  $\varepsilon > 0$  the set  $\{i \in I \mid |r_i - r| < \varepsilon\}$  is in  $D$ . Suppose that for each  $i$ ,  $(X_i, d_i)$  is a metric space with a distinguished point  $a_i$ . Let

$$X_\infty = \left\{ (x_i) \in \prod_i X_i \mid \sup_{i \in I} d(x_i, a_i) < \infty \right\} / \sim,$$

where  $(x_i) \sim (y_i)$  if and only if  $\lim_{i, D} d(x_i, y_i) = 0$ . Let  $d_\infty$  be the metric on  $X_\infty$  defined by  $d_\infty((x_i), (y_i)) = \lim_{i, D} d(x_i, y_i)$ . Leaving the dependence on the choice of the base points  $a_i$  implicit, we will call this an *ultraproduct* of the metric spaces  $(X_i, d_i)$ , denoted by  $(\prod_i (X_i, d_i))_D$ . If there is a uniform bound on the diameters of these spaces, the choice of the sequence  $(a_i)$  of “anchor points” is clearly irrelevant.

This ultraproduct construction is an instance of Luxemburg’s nonstandard hull construction [33]. We can extend it to ultraproducts of a sequence  $(X_i)$  of normed spaces using  $a_i = 0$  and the distance given by the norm. Ultraproducts of Banach spaces were introduced by Dacunha-Castelle and Krivine [10], and are an important tool in a number of branches of analysis (see e.g. [16]).

In first-order model theory, one can take an ultraproduct of any sequence of structures  $M_i$ , and Los’s theorem says that any first-order sentence  $\varphi$  is true in the ultraproduct if and only if it is true in almost every  $M_i$ , in the sense of  $D$ ; in other words, if and only if  $\{i \mid M_i \models \varphi\} \in D$ . The constructions above, however, are not ultraproducts in the first-order sense, since we restrict to “finite” elements, mod out by infinitesimal proximity  $\sim$ , and (implicitly, by taking limits with respect to

$D$ ) pass to the standard part of nonstandard distances and norms. This gives rise to two complications.

First, if we extend the metric or normed spaces with other functions, their lifting to the ultraproduct will not be well defined if they fail to map finite elements to finite elements, or fail to respect  $\sim$ . We can lift, however, any family  $(f_i)$  of functions that satisfies an appropriate *uniform boundedness condition* (roughly, elements of the family are uniformly bounded on bounded sets around the base point) and an appropriate *uniform continuity condition* (which is to say that there is a uniform modulus of uniform continuity on such sets). The resulting function on the ultraproduct will be denoted  $(\prod_i f_i)_D$ . For details, see [16, Section 4] or [7, Section 4].

Second, Los's theorem needs to be modified. One strategy, described in [16], is to restrict attention to a class of *positively bounded formulas*. These are formulas generated from atomic formulas  $r \leq t$  and  $t \leq r$ , where  $t$  is an appropriate term and  $r$  is rational, using only the positive connectives  $\wedge$  and  $\vee$ , as well as universal and existential quantification over compact balls in the structure. An *approximation* to such a formula is obtained by replacing each  $r$  in an atomic formula  $r \leq t$  by any  $r' < r$ , and each  $r$  in an atomic formula  $t \leq r$  by any  $r' > r$ . Say that a formula  $\varphi$  with parameters is *approximately true* in a structure if every approximation  $\varphi'$  to  $\varphi$  is true in the structure. One can then show that if  $a_1, \dots, a_n$  are elements of the ultraproduct with each  $a_j$  represented by the sequence  $(a_{j,i})_{i \in I}$ , then a positively bounded formula  $\varphi(a_1, \dots, a_n)$  is approximately true in the ultraproduct  $(\prod_i \mathcal{M}_i)_D$  if and only if

$$\{i \mid \mathcal{M}_i \models \varphi'(a_{1,i}, \dots, a_{n,i})\} \in D$$

for every approximation  $\varphi'$  to  $\varphi$ .

Suppose  $\Gamma$  is a set of positively bounded sentences, and  $C$  is the class of structures  $\mathcal{M}$  that approximately satisfy each sentence in  $\Gamma$ . The previous equivalence implies that  $C$  is closed under ultraproducts. In fact, Henson and Iovino [16, Proposition 13.6] show that a class of structures  $C$  can be axiomatized in this way if and only if  $C$  is closed under isomorphisms, ultraproducts, and ultraroots.

Another strategy, described in [7], is to modify first-order semantics so that formulas take on truth values in a bounded interval of reals, in which case the truth value of a formula  $\varphi$  in the ultraproduct is the  $D$ -limit of its truth values in the individual structures. Spelling out the details here would take us too far afield. Below we will only use the fact that certain classes of structures and hypotheses are preserved under ultraproducts, as well as the easy fact that a quantifier-free positively bounded formula  $\varphi$  is true in a structure if and only if every approximation to it is true, thereby simplifying the equivalence above.

### 3. EXAMPLES

Let  $T$  be any nonexpansive operator on a Hilbert space,  $\mathcal{H}$ , let  $f$  be any element of  $\mathcal{H}$ , and for each  $N \geq 1$  let  $A_N f$  denote the ergodic average  $\frac{1}{N} \sum_{n < N} T^n f$ . Riesz's generalization of von Neumann's mean ergodic theorem states that the sequence  $(A_N f)$  of averages converges in the Hilbert space norm. The following generalization is due to Lorch [32], but also a consequence of results of Riesz [37], Yosida [49], and Kakutani [20] from around the same time (see [28, p. 73]). A linear operator  $T$  on a Banach space  $\mathcal{B}$  is *power bounded* if there is an  $M$  such that  $\|T^n\| \leq M$  for every  $n$ .

**Theorem 3.1.** *If  $T$  is any power-bounded linear operator on a reflexive Banach space  $\mathcal{B}$ , and  $f$  is any element of  $\mathcal{B}$ , then the sequence  $(A_N f)_{N \in \mathbb{N}}$  converges.*

As noted in Section 1, even in the original von Neumann setting there is no uniform bound on the rate of convergence. Moreover, a rate of convergence is generally not computable from the given data [6, 45]; see also the discussion in [4, Section 5]. However, we can obtain a strong uniformity if we shift attention to metastability.

**Theorem 3.2.** *Let  $C$  be any class of Banach spaces with the property that the ultraproduct of any countable collection of elements of  $C$  is a reflexive Banach space. For every  $\rho > 0$ ,  $M$ , and function  $F : \mathbb{N} \rightarrow \mathbb{N}$ , there is  $K$  such that the following holds: given any Banach space  $\mathcal{B}$  in  $C$ , any linear operator on  $\mathcal{B}$  satisfying  $\|T^n\| \leq M$  for every  $n$ , any  $f \in \mathcal{B}$ , and any  $\varepsilon > 0$ , if  $\|f\|/\varepsilon \leq \rho$ , then there is an  $n \leq K$  such that  $\|A_i f - A_j f\| < \varepsilon$  for every  $i, j \in [n, F(n)]$ .*

*Proof.* Scaling, we can restrict attention to elements  $f$  such that  $\|f\| \leq 1$ . Fix  $C$ , and suppose the claim is false for  $\rho = 1/\varepsilon$  and  $F$ . For each  $k$  in  $\mathbb{N}$ , choose a counterexample, that is, a Banach space  $\mathcal{B}_k \in C$ , a linear operator  $T_k$  such that  $\|T_k^n\| \leq M$  for every  $n$ , and an element  $f_k$  such that  $\|f_k\| \leq 1$  and for every  $n \leq k$  there are  $i, j \in [n, F(n)]$  such that  $\|A_i f_k - A_j f_k\| \geq \varepsilon$ .

The fact that each  $\|T_k\| \leq M$  for every  $k$  guarantees that the family  $(T_k)$  satisfies the uniform boundedness and uniform continuity conditions. Let  $D$  be a nonprincipal ultrafilter on  $\mathbb{N}$ , and let  $\mathcal{B} = (\prod_k \mathcal{B}_k)_D$  be the Banach space ultraproduct, with  $T = (\prod_k T_k)_D$  and  $f = (\prod_k f_k)_D$ . By hypothesis,  $\mathcal{B}$  is reflexive. We have  $\|T^n\| \leq M$  for every  $n$ , since this is true of each  $T_k$ . Moreover, for every  $n$ ,  $\|A_i f - A_j f\| \geq \varepsilon$  for some  $i, j \in [n, F(n)]$ , since this is true of the elements  $f_k$  in all but finitely many of the structures  $\mathcal{B}_k$ . This contradicts Theorem 3.1.  $\square$

The class  $C$  of *all* reflexive Banach spaces does not satisfy the hypothesis of Theorem 3.2, which is to say, an ultraproduct of reflexive Banach spaces need not be reflexive. However, there are interesting classes  $C$  to which the theorem applies. For example, every uniformly convex Banach space is reflexive, and if one fixes a modulus of uniform convexity, the class of uniformly convex spaces with that modulus is closed under ultraproducts. Thus, Theorem 3.2 guarantees the existence of a uniform bound on the rate of metastability that depends only on  $\rho$ ,  $M$ ,  $F$ , and the modulus of uniform convexity.

For another example, say that a Banach space  $\mathcal{B}$  is  $J$ -( $n, \varepsilon$ ) *convex* if for every  $x_1, \dots, x_n$  in the unit ball of  $\mathcal{B}$  there is a  $j$ ,  $1 \leq j \leq n$ , such that

$$\left\| \sum_{i < j} x_i - \sum_{i \geq j} x_i \right\| \leq n(1 - \varepsilon).$$

A space is  $J$ -convex if and only if it is  $J$ -( $n, \varepsilon$ ) convex for some  $n \geq 2$  and  $\varepsilon > 0$ . Pisier [34] shows that a Banach space is  $J$ -convex if and only if it is super-reflexive, so, in particular, every  $J$ -convex space is reflexive. Moreover, it is immediate from the form of the definition that, for fixed  $n \geq 2$  and  $\varepsilon > 0$ , the class of  $J$ -( $n, \varepsilon$ ) convex Banach spaces is closed under ultraproducts. Thus, Theorem 3.2 once again guarantees the existence of a uniform bound on the rate of metastability that depends only on  $\rho$ ,  $M$ ,  $F$ ,  $n$ , and  $\varepsilon$ . Note that for  $n = 2$ , a space is  $J$ -( $n, \varepsilon$ ) convex for some

$\varepsilon > 0$  if and only if it is *uniformly non-square*, a weakening of strict convexity due to James [18].

The list of classes of structures to which Theorem 3.2 applies can easily be extended. For example, we can obtain many classes of spaces that satisfy the hypothesis of that theorem by simply fixing bounds on appropriate parameters in the various characterizations of superstability given by Pisier in Chapter 3 of [34]. Other examples of classes of reflexive spaces that are closed under formation of ultraproducts can be found in [30, 36, 35].

In the case of a nonexpansive map on a uniformly convex Banach space, Avigad and Rute [5] provide an explicit uniform bound on the number of  $\varepsilon$ -fluctuations of the sequence  $(A_n f)$ , in terms of  $\rho$  and the modulus of uniform convexity. In the case of a nonexpansive map on a Hilbert space, Jones, Ostrovskii, and Rosenblatt [19] provide an even stronger result, in the form of an explicit square-function inequality for the sequence  $(A_n f)$ . We do not know the extent to which these stronger uniformities extend. (Safarik and Kohlenbach [39] provide some general conditions that guarantee that it is possible to compute a bound on the number of  $\varepsilon$ -fluctuations.)

As noted in the introduction, the method of proving Theorem 3.2 is quite general, applying to any class of structures and set of hypotheses that are preserved under ultraproducts. (Restricting to elements  $f$  with  $\|f\| \leq 1$  is needed to ensure that the ultraproduct of such elements gives rise to an element of the ultraproduct.) In fact, any it is enough to know that *some* ultraproduct satisfies the relevant hypothesis, allowing flexibility in the choice of ultrafilter. Rather than state all this formally, we will illustrate with two additional examples, with respect to which notions of metastability have been considered in the past.

For the first example, we consider extensions of the mean ergodic theorem to “diagonal averages.” Furstenberg’s celebrated ergodic-theoretic proof of Szemerédi’s theorem involves averages of the form

$$\frac{1}{n} \sum_{i < n} f_1(T_1^{-i} x) \cdots f_j(T_j^{-i} x)$$

where  $T_1, \dots, T_j$  are commuting measure-preserving transformations of a finite measure space  $(X, \mathcal{X}, \mu)$ . Settling a longstanding open problem, Tao [42] showed that such sequences always converge in the  $L^2(X)$  norm. This result was recently generalized by Walsh [46], as follows:

**Theorem 3.3.** *Let  $(X, \mathcal{X}, \mu)$  be a finite measure space with a measure-preserving action of a nilpotent group  $G$ . Let  $T_1, \dots, T_l$  be elements of  $G$ , and let*

$$(p_{i,j})_{i=1,\dots,l; j=1,\dots,d}$$

*be a sequence of integer-valued polynomials on  $\mathbb{Z}$ . Then for any  $f_1, \dots, f_d \in L^\infty(X, \mathcal{X}, \mu)$ , the sequence of averages*

$$\frac{1}{N} \sum_{n=1}^N \prod_{j=1}^d \left( T_1^{p_{1,j}(n)} \cdots T_l^{p_{l,j}(n)} \right) f_j$$

*converges in the  $L^2(X)$  norm.*

When the relevant data  $\vec{T}, \vec{p}$  are clear, it will be convenient to write  $A_N(\vec{f})$  for these averages. Once again, a compactness argument yields the following uniformity:

**Theorem 3.4.** *For every  $r, l, d, s, \rho > 0$ , and function  $F : \mathbb{N} \rightarrow \mathbb{N}$ , there is a  $K$  such that the following holds: given a nilpotent group  $G$  of nilpotence class at most  $r$ , elements  $T_1, \dots, T_l$  in  $G$ , a sequence  $(p_{i,j})_{i=1, \dots, l; j=1, \dots, d}$  of integer-valued polynomials on  $\mathbb{Z}$  of degree at most  $s$ , a probability space  $(X, \mathcal{X}, \mu)$ , measure-preserving action of  $G$  on  $(X, \mathcal{X}, \mu)$ , and any sequence of elements  $f_1, \dots, f_d \in L^\infty(X, \mathcal{X}, \mu)$ , if  $\|f_i\|_\infty/\varepsilon \leq \rho$  for each  $i$ , then there is an  $n \leq K$  such that  $\|A_i(\vec{f}) - A_j(\vec{f})\| < \varepsilon$  for every  $i, j \in [n, F(n)]$ .*

As above, we can restrict attention to the case where  $\|f_i\|_\infty \leq 1$  in the statement of the theorem, and without loss of generality we can assume that  $G$  is generated by  $T_1, \dots, T_l$ . An ultraproduct construction due to Loeb [31], analogous to the constructions described in Section 1, can be used to amalgamate a sequence of measure spaces  $(X_k, \mathcal{X}_k, \mu_k)$  to a measure space  $(X, \mathcal{X}, \mu)$ , and since first-order properties of discrete structures are preserved under ultraproducts, the ultraproduct of a sequence  $(G_k)$  of groups of nilpotence class at most  $r$  is again a group of nilpotence class at most  $r$ . A measure-preserving action of each  $G_k$  on  $(X_k, \mathcal{X}_k, \mu_k)$  gives rise to a measure-preserving action of  $G$  on  $(X, \mathcal{X}, \mu)$ , and the product of the spaces  $L^2(X_k, \mathcal{X}_k, \mu_k)$  embeds isometrically into the space  $L^2(X, \mathcal{X}, \mu)$  (see, for example, [15, Section 5]).

There is a catch, though: the ultraproduct of a sequence of polynomials  $p_k$  with coefficients in  $\mathbb{Z}$  need not be a polynomial, since the coefficients can “go off to infinity.” One could rule that out by assuming that there is a uniform bound on those coefficients, in which case the value  $K$  in the statement of the theorem would depend on that bound as well. As it turns out, however, in this particular case there is a trick that eliminates the dependence on this parameter. Call a sequence  $(g_n)$  of elements of the form  $g_n = T_1^{p_1(n)} \dots T_l^{p_l(n)}$  a *polynomial sequence*.

**Lemma 3.5.** *Let  $G$  be a nilpotent group, and let  $(g_n)$  be a polynomial sequence of elements of  $G$  as above. Then there is a nilpotent extension  $\eta : \hat{G} \rightarrow G$  and elements  $\tau$  and  $c$  such that for every  $n$ ,  $g_n = \eta(\tau^n c)$ . Moreover, there is a bound on the nilpotence class of  $\hat{G}$  that depends only on bounds on the nilpotence class of  $G$ , the number  $l$  of polynomials, and a bound on their degrees.*

Via  $\eta$ , the action of  $G$  on  $X$  lifts to an action of  $\hat{G}$  on  $X$ , whereby the action of  $g_n$  lifts to the action of  $\tau^n c$ . Applying the lemma  $d$  times, we can thus assume that each polynomial sequence  $g_{i,n} = T_1^{p_{i,1}(n)} \dots T_l^{p_{i,l}(n)}$  appearing in the statement of Walsh’s theorem is of the form  $\tau_i^n c_i$  for some  $\tau_i$  and  $c_i$  in  $G$ , at the expense of increasing the nilpotence rank of  $G$ .

Lemma 3.5 is a special case of a construction carried out by Leibman [29] in the more general setting of an action of Lie group, with both continuous and discrete elements. We are grateful to Terence Tao for bringing this lemma to our attention, and pointing out that it can be used to obtain a stronger uniformity in the statement of Theorem 3.4. As Leibman points out, an instance of this trick was used by Furstenberg [11, page 31]. Leibman’s construction can be divided into two parts: Proposition 3.14 of [29] shows how to define a nilpotent extension  $\eta : \tilde{G} \rightarrow G$ , a unipotent automorphism  $\tau$  of  $\tilde{G}$ , and an element  $c$  of  $\tilde{G}$ , such that for every  $n$ ,  $g(n) = \eta(\tau^n(c))$ ; and Proposition 3.9 shows that the extension  $\hat{G}$  of  $\tilde{G}$  by  $\tau$  is again a nilpotent group. Here, saying that  $\tau$  is a unipotent automorphism means that the mapping  $\xi(a) = \tau(a)a^{-1}$  has the property that  $\xi^q$  is the identity for sufficiently large  $q$ . The proof of Proposition 3.9 gives an explicit bound on how large  $q$  has to

be and the nilpotence class of  $\tilde{G}$ ; and Proposition 1 of Gruenberg [14] then provides the requisite bound on the nilpotence class of  $\hat{G}$ .

With this lemma in hand, we can prove Theorem 3.4.

*Proof.* As above, we can restrict attention to the case where  $\|f_i\|_\infty \leq 1$  in the statement of the theorem. Using Lemma 3.5, we can moreover assume  $d = 2l$ ,  $s = 1$ , and for every  $i$ ,  $p_{i,2i}(n) = n^i$ ,  $p_{i,2i+1}(n) = 1$ , and  $p_{i,j} = 0$  for all other  $j$ , so that the  $i$ th polynomial sequence is given by  $g_{i,n} = T_{2i}^n T_{2i+1}$ . Given  $\rho > 0$  and  $F$  such that the claim is false, for each  $k$  choose a probability space  $(X_k, \mathcal{X}_k, \mu_k)$ , a group  $G_k$  of nilpotence class at most  $r$ , and elements  $T_{1,k}, \dots, T_{l,k}$ , and elements  $f_{1,k}, \dots, f_{d,k}$  with infinity norm at most 1 such that for every  $n \leq k$ ,  $\|A_i(\vec{f}_k) - A_j(\vec{f}_k)\| \geq \varepsilon$  for some  $i, j \in [n, F(n)]$ .

Fix a nonprincipal ultrafilter  $D$  on  $\mathbb{N}$ . Let  $(X, \mathcal{X}, \mu)$  be the result of applying the Loeb construction to the sequence of spaces  $(X_k, \mathcal{X}_k, \mu_k)$ , let  $G$  be the ultraproduct of the sequence  $(G_k)$  with respect to  $D$ , and for each  $i$ , let  $T_i = (\prod_k T_{i,k})_D$ . Then  $G$  has nilpotence class at most  $r$ , and each  $T_i$  is measure-preserving transformation of  $X$ . But then, as in the proof of Theorem 3.2, the elements  $f_1 = (\prod_k f_{1,k})_D, \dots, f_d = (\prod_k f_{d,k})_D$  yield a counterexample to Theorem 3.3.  $\square$

Tao [40] shows that one can alternatively formulate Walsh's theorem in algebraic terms, which allows one to avoid the reference to the Loeb construction in the proof of Theorem 3.4. In fact, both Walsh's original proof [46] and Tao's later proof of Walsh's result [40] establish Theorem 3.4 directly. Tao's proof of his prior result [42] also established the corresponding uniformity, but there are now other proofs of that theorem that do not [2, 17, 44]. Tao [40] emphasizes that Theorem 3.4 is stronger than Theorem 3.3; the observation here is that they are essentially the same, modulo compactness and Lemma 3.5.

We consider a final example, this time from nonlinear ergodic theory. Fix a Hilbert space  $\mathcal{H}$ . Let  $C$  be a bounded, closed, convex subset of  $\mathcal{H}$ , and let  $T$  be a nonexpansive map from  $C$  to  $C$ . Let  $(\lambda_n)$  be a sequence of elements of  $[0, 1]$ , and let  $f$  and  $u$  be any elements of  $C$ . The *Halpern iteration* corresponding to  $T$ ,  $(\lambda_n)$ ,  $f$ , and  $u$  is the sequence given by

$$f_0 = f, \quad f_{n+1} = \lambda_{n+1}u + (1 - \lambda_{n+1})Tf_n.$$

If  $T$  is linear,  $u = f$ , and  $\lambda_n = 1/(n+1)$ , then  $(f_n)$  is the familiar sequence  $(A_n f)$  of ergodic averages. Wittmann [47] showed that, assuming the set of fixed points of  $T$  is nonempty, the following conditions on the sequence  $(\lambda_n)$  suffice to ensure that the sequence  $f_n$  of Halpern iterates converges to the projection onto the space of fixed points:

- $\lim_{n \rightarrow \infty} \lambda_n = 0$
- $\sum_{n=1}^{\infty} \|\lambda_{n+1} - \lambda_n\|$  converges
- $\sum_{n=1}^{\infty} \lambda_n = \infty$ .

In particular, these are satisfied when  $\lambda_n = 1/(n+1)$ .

The linear structure of  $\mathcal{H}$  only comes into play in the assumption that  $C$  is convex. Seajung [38] has generalized Wittmann's result to CAT(0) spaces, which are metric spaces with an abstract notion of "linear combination," that is, metric spaces equipped with a function  $W(x, y, \lambda)$  which, intuitively, plays the role of  $(1 - \lambda)x + \lambda y$ . The specific axioms that  $W$  is assumed to satisfy can be found in [9, 25, 38]; we only need the fact, established in [9, pages 77–78], that the

ultraproduct of  $\text{CAT}(0)$  spaces is again a  $\text{CAT}(0)$  space. Saejung's theorem states the following:

**Theorem 3.6.** *Let  $C$  be a closed convex subspace of a complete  $\text{CAT}(0)$  space, and let  $T : C \rightarrow C$  be a nonexpansive map such that the set of fixed points of  $T$  is nonempty. Suppose  $(\lambda_n)$  satisfies the three conditions above. Then for any  $u, f$  in  $C$ , the sequence of Halpern iterates  $(f_n)$  converges to the projection of  $u$  onto the set of fixed points of  $T$ .*

Kohlenbach and Leuştean [25] have shown that there is a uniform bound on the rate of metastability, given by a primitive recursive functional. If one is only interested in uniformity and not the particular rate, the following provides a quick proof:

**Theorem 3.7.** *Fix  $(\lambda_n)$  satisfying (1–3) above. For every  $\varepsilon > 0$ ,  $M$ , and function  $F : \mathbb{N} \rightarrow \mathbb{N}$ , there is a  $K$  such that the following holds: given a  $\text{CAT}(0)$  space  $(X, d, W)$ , a closed convex subset  $C$  of  $X$  with diameter at most  $M$ , a nonexpansive map  $T : C \rightarrow C$  with a fixed point in  $C$ , and  $f, u$  in  $C$ , if  $(f_n)$  denotes the sequence of Halpern iterates, then there is an  $n \leq K$  such that  $d(f_i, f_j) < \varepsilon$  for every  $i, j$  in  $[n, F(n)]$ .*

*Proof.* Once again, use an ultraproduct construction to amalgamate a sequence of purported counterexamples. We have already noted that the ultraproduct of  $\text{CAT}(0)$  spaces is again a  $\text{CAT}(0)$  space. The uniform bound on the diameter of each of the sets  $C$  is also a bound on the diameter of their product. The fact that convexity is preserved is immediate; and it is well known that an ultraproduct of closed sets is again closed (see, for example, [7, Proposition 5.3]).  $\square$

Theorem 3.7 can also be seen as a consequence of Corollary 4.25 in Gerhardy and Kohlenbach [12], modulo verification of the fact that Saejung's theorem can be derived in the formal axiomatic system mentioned there. As Gerhardy and Kohlenbach note, one can weaken the hypothesis that  $T$  has a fixed point in  $C$  to the hypothesis that  $T$  has an  $\varepsilon$ -fixed point in  $C$  for every  $\varepsilon > 0$ . This is easy to see from the ultraproduct argument as well, since the ultralimit  $\varepsilon$ -fixed points for a sequence  $\varepsilon$  decreasing to 0 is an actual fixed point. This fact is commonly used in applications of ultraproducts to fixed-point theory; see, for example, Aksoy and Khamsi [1].

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